Technical Note: Walsh-Aligned Diagonal Modulation as Structured Permutation

A useful way to understand the behavior of HD is to view Walsh functions as products of Rademacher functions. In this viewpoint, the Walsh functions form a closed binary harmonic system under multiplication.

When the diagonal matrix D contains a Walsh function on its diagonal, multiplying H by D does not create a fundamentally new mixing pattern. Instead, it simply permutes the existing Walsh basis structure already present in the Walsh–Hadamard matrix.

This happens because multiplying one Walsh function by another produces a third Walsh function from the same family. As a result:

- The ordering of the Walsh functions in H is changed,
- the global oscillatory structure remains coherent,
- and the overall system retains strong internal symmetry.

From a spectral viewpoint, Walsh-aligned D preserves the intrinsic mode structure of the transform. The operation behaves more like a structured relabeling or binary phase shift than a genuinely new interaction operator.

The interesting behavior appears when D is only approximately Walsh-aligned. In that case, the exact closure property is broken slightly, allowing weak coupling between Walsh modes. Under iteration this produces:

- visible oscillations,
- interference-like beating patterns,
- mode leakage,
- and slowly evolving internal structures.

This explains why Walsh-aligned systems often appear highly ordered and stable, while small perturbations reveal hidden oscillatory dynamics that are otherwise suppressed by the underlying symmetry.

Additional Note: Thue–Morse Structure and Walsh Alignment

An especially interesting example is the Thue–Morse sequence, which can be viewed as one of the Walsh functions — specifically the product of all Rademacher functions for a finite Walsh system size.

In practical terms, the Thue–Morse sequence is generated simply by taking the parity of the binary representation of an integer counter:

- even bit parity → +1
- odd bit parity → -1

This gives a highly oscillatory but perfectly Walsh-aligned sign pattern. Because it belongs exactly to the Walsh family, using it as the diagonal modulation D)preserves the internal Walsh symmetry structure and acts as a structured permutation of the Walsh basis.

This makes the Thue–Morse sequence an interesting “maximally oscillatory yet perfectly aligned” example inside the Walsh system.

A contrasting situation occurs when the integer sequence itself is permuted before parity is taken. For example:

- Weyl addition sequences,
- additive irrational rotations,
- or other low-discrepancy permutations,

destroy the exact Walsh alignment.

In that case, the resulting parity pattern no longer corresponds cleanly to any Walsh function. The closure property under Walsh multiplication is broken, and the system can no longer reduce to simple basis permutation.

This produces:

- imperfect spectral alignment,
- mode leakage,
- richer interference structure,
- and visible oscillatory dynamics under iteration.

Conceptually, this creates a transition between:

- exact binary harmonic symmetry (Walsh-aligned),
- and quasi-structured but non-Walsh modulation (misaligned parity systems).

That boundary appears to be particularly rich dynamically, because the system remains highly structured while no longer possessing exact Walsh closure.